JOURNAL OF FUNCTIONAL ANALYSIS 45, 95–108 (1982)

The Algebra of Bounded Holomorphic Martingales

KEITH CARNE

Rutgers University, New Brunswick, New Jersey 08903

Communicated by the Editors

Received June 1981

The algebra of bounded holomorphic martingales introduced by N. Th. Varopoulos (*J. Funct. Anal.*, in press) is studied. In particular, it is shown that this is a logmodular algebra with a spectrum having no analytic structure.

1. INTRODUCTION

The purpose of this article is to study the structure of the algebra of bounded holomorphic martingales introduced by N. Th. Varopoulos [5]. This algebra shares many of the properties of the classical algebra $H_\infty(D)$ of bounded analytic functions on the unit disc D and its interest stems largely from the possibility of using it to study $H_\infty(D)$. However, it transpires that it is an interesting example of an uniform algebra in its own right. Its most notable property is that it is a logmodular algebra with a spectrum bearing no analytic structure, as is shown in Theorem 3 below.

In the following section the structure of the spectrum is investigated briefly, to show how the probability theory is reflected in the algebraic properties. Section 4 contains the proof that the spectrum has no analytic structure. Finally we indicate how the algebra is related to a certain algebra of germs of analytic functions.

For simplicity we shall consider the standard complex Brownian motion although the material in all but the last section will generalize to the case of several complex variables. Let $(z_t : t \geqslant 0)$ be the standard complex Brownian motion starting at O on the probability space (Ω, P). The σ-field $\mathscr{F}_t$ is that generated by z_s for $s \leqslant t$, and $\mathscr{F}$ is the σ-field generated by them all. Consider a process

$$F: \Omega \times [0, \infty) \to \mathbf{C}; \qquad (\omega, t) \to F_t(\omega).$$

For any stopping time T we will denote by F^T the stopped process

$$F^T: (\omega, t) \to f_{t \wedge T(\omega)}(\omega).$$

0022-1236/82/010095–14$02.00/0

The stopping time

$$Q(\omega) = \inf(t: |z_t(\omega)| \geqslant 1)$$

at which the Brownian motion quits D will arise frequently. Those processes which can be represented as a stochastic integral of the form

$$F_t = F_0 + \int_0^t \partial F \, dz$$

for some previsible process ∂F will be called *holomorphic* and ∂F will be called the *derivative* of F.

For each $1 \leqslant p \leqslant \infty$ the F-measurable random variables which are p-power integrable form a Banach space $L_p(\mathscr{F})$. For each $F \in L_p(\mathscr{F})$, the conditional expectations $E(F \mid \mathscr{F}_t)$ form a martingale and we shall choose representatives F_t for $E(F \mid \mathscr{F}_t)$ so that

$$F.(\omega): t \to F_t(\omega)$$

is continuous for each $\omega \in \Omega$. Then F is identified with the process $(\omega, t) \to F_t(\omega)$. The holomorphic martingales in $L_p(\mathscr{F})$ form a subspace which we denoted by $H_p(\mathscr{F})$ (cf. [5]).

Under pointwise multiplication $L_\infty(\mathscr{F})$ is a Banach algebra and the Gelfand transform:

$$\hat{}: L_\infty(\mathscr{F}) \to C(\operatorname{Spec} L_\infty(\mathscr{F}))$$

is an isometry. Let 1_u be the characteristic function of U. Then each $x \in \operatorname{Spec} L_\infty(\mathscr{F})$ corresponds to the ultrafilter

$$\{U \in \mathscr{F}: \hat{1}_u(x) = 1\}$$

on $\mathscr{F}$ modulo the null sets. Conversely, every such ultrafilter is derived from a point of the spectrum. Thus we will identify x with the ultrafilter. N. Th. Varopoulos remarked in [5] that Itô's lemma implied that $H_\infty(\mathscr{F})$ is a subalgebra of $L_\infty(\mathscr{F})$, and indeed a *logmodular* algebra.

I am glad to acknowledge my gratitude to N. Th. Varopoulos for introducing me to his work on holomorphic martingales and suggesting the study of the spectrum of $H_\infty(\mathscr{F})$.

2. The Spectrum of $H_\infty(\mathscr{F})$

If $F, G \in H_\infty(\mathscr{F})$, then Itô's lemma shows that $F \cdot G$ is holomorphic with

$$\partial(F \cdot G) = F \cdot \partial G + \partial F \cdot G.$$

Hence $H_\infty(\mathscr{F})$ is a subalgebra of $L_\infty(\mathscr{F})$ for pointwise multiplication (see [5, Theorem 3.1]). In particular,

$$E(F \cdot G \mid \mathscr{F}_t) = E(F \mid \mathscr{F}_t) \cdot E(G \mid \mathscr{F}_t)$$

for each $t \in [0, \infty]$. This shows that for any random time $R\colon \Omega \to [0, \infty]$ the map

$$J_R : H_\infty(\mathscr{F}) \to L_\infty(\mathscr{F}),$$
$$J_R F\colon \omega \to F_{R(\omega)}(\omega)$$

is an algebra homomorphism. If $x \in \operatorname{Spec} L_\infty(\mathscr{F})$ then the composite

$$H_\infty(\mathscr{F}) \xrightarrow{J_R} L_\infty(\mathscr{F}) \xrightarrow{x} \mathbf{C}$$

is a character of $H_\infty(\mathscr{F})$ we will denote by (x, R). We shall be particularly concerned with these characters when R is a stopping time, or a fixed time.

The algebra $H_\infty(\mathscr{F})$ was introduced by N. Th. Varopoulos who showed that it shares many of the properties of $H_\infty(D)$. In particular, he proved the following analogue of the corona theorem [5, Theorem 8.1]:

If $(F_n : n = 1, \dots, N)$ from $H_\infty(\mathscr{F})$ satisfy

$$\sum |E(F_n \mid \mathscr{F}_t)| \geqslant \delta \qquad (*)$$

almost surely for each $t \geqslant 0$, then there exist $G_n \in H_\infty(\mathscr{F})$ with $\sum F_n \cdot G_n = 1$.

Condition (*) is equivalent to

$$\sum |\hat{F}_{nt}(x)| \geqslant \delta$$

for all $x \in \operatorname{Spec} L_\infty(\mathscr{F})$, so we see that the characters

$$(x, t)\colon H_\infty(\mathscr{F}) \to \mathbf{C},$$
$$F \to \hat{F}_t(x)$$

are dense in $\operatorname{Spec} H_\infty(\mathscr{F})$ for the weak * topology. Thus any character χ of $H_\infty(\mathscr{F})$ is the limit of some net (x_α, t_α). We shall show that if we allow the t_α to be stopping times T_α rather than fixed times then χ is the limit of a net (x, T_α) for one *fixed* x.

For $F \in H_\infty(\mathscr{F})$, the *maximal function* F^* is

$$F^*\colon \omega \to \sup |F_t(\omega)|.$$

Fix $x \in \operatorname{Spec} L_\infty(\mathscr{F})$. Then

$$(F \cdot G)^{*\hat{}}(x) \leqslant F^{*\hat{}}(x) \cdot G^{*\hat{}}(x)$$

so

$$J_x = \{F \in H_\infty(\mathscr{F}): F^{*\hat{}}(x) = 0\}$$

is a closed ideal of $H_\infty(\mathscr{F})$. If we let A_x be the Banach algebra $H_\infty(\mathscr{F})/J_x$ and identify its spectrum with a subset of $\operatorname{Spec} H_\infty(\mathscr{F})$, then we have the following result.

THEOREM 1. *For each x, the algebra A_x is an uniform algebra with spectrum*

$$\Sigma_x = \{\chi \in \operatorname{Spec} H_\infty(\mathscr{F}): |\chi(F)| \leqslant F^{*\hat{}}(x) \text{ for all } F \in H_\infty(\mathscr{F})\}.$$

The characters

$$\{(x, T): T \text{ a stopping time}\}$$

are dense in Σ_x.

Every character of $H_\infty(\mathscr{F})$ lies in at least one Σ_x.

Proof. It is clear that the norm in A_x of an equivalence class $[F]$ satisfies

$$\begin{aligned} \|[F]\| &= \inf(\|F - G\|_\infty : G \in J_x) \\ &\geqslant \inf(|F^{*\hat{}}(x) - G^{*\hat{}}(x)|: G \in J_x) \geqslant F^{*\hat{}}(x). \end{aligned}$$

Conversely, if $\lambda > F^{*\hat{}}(x)$, then

$$U = \{\omega \in \Omega: F^*(\omega) < \lambda\} \in x,$$

and the stopping time

$$T(\omega) = \inf(t: |F_t(\omega)| \geqslant \lambda)$$

is finite on U. Thus $G = F - F_T$ has $G^{*\hat{}}(x) = 0$ and

$$\|F - G\|_\infty = \|F_T\|_\infty \leqslant \lambda.$$

This shows that

$$\|[F]\| = F^{*\hat{}}(x)$$

so Σ_x is the spectrum of A_x.

If T is any stopping time (or indeed any random time) then $(x, T) \in \Sigma_x$. Suppose that $\lambda < \|[F]\| = F^{*\wedge}(x)$. Then, as above, the stopping time

$$T(\omega) = \inf(t: |F_t(\omega)| \geqslant \lambda)$$

is finite on

$$V = \{\omega \in \Omega: F^*(\omega) > \lambda\} \in x.$$

Therefore, $|\hat{F}_T(x)| \geqslant \lambda$ and consequently A_x is a uniform algebra.

To prove that $\{(x, T)\}$ is dense in Σ_x it is sufficient to show that whenever $[F_n] \in A_x$ satisfy

$$\sum |\hat{F}_{nT}(x)| \geqslant \delta > 0 \qquad \text{for all } T,$$

then there exist $[G_n] \in A_x$ with $\sum [F_n][G_n] = 1$. To this end, define

$$S(\omega) = \inf\left(t: \sum |F_{nt}(\omega)| \leqslant \tfrac{1}{2}\delta\right).$$

This is a stopping time with

$$U = \{\omega: S(\omega) = \infty\} \in x.$$

Let $H_n = F_{nS} \in H_\infty(\mathscr{F})$. Then $[F_n] = [H_n]$ and

$$\sum |H_{nt}(\omega)| \geqslant \tfrac{1}{2}\delta > 0$$

almost surely for each $t \geqslant 0$. Now the corona theorem for $H_\infty(\mathscr{F})$ implies that there exists $G_n \in H_\infty(\mathscr{F})$ with $\sum F_n G_n = 1$. Consequently,

$$\sum [F_n][G_n] = \sum [H_n][G_n] = 1$$

as required.

It remains to show that each $\chi \in \operatorname{Spec} H_\infty(\mathscr{F})$ lies in at least one Σ_x. There is a net (x_α, t_α) converging to χ and we may choose x as any limit point of the net (x_α) in $\operatorname{Spec} L_\infty(\mathscr{F})$. Take $F \in H_\infty(\mathscr{F})$ and $\lambda > F^{*\wedge}(x)$. Then

$$U = \{\omega \in \Omega: F^*(\omega) < \lambda\}$$

is in x and therefore in x_α for arbitrarily large α. For such α,

$$|\hat{F}_{t_\alpha}(x)| \leqslant \lambda$$

so

$$|\chi(F)| = \operatorname{Lim} |\hat{F}_{t_\alpha}(x)| \leqslant \lambda.$$

This shows that $\chi \in \Sigma_x$. ∎

In general, a character of $H_\infty(\mathscr{F})$ does not lie in a unique Σ_x; for example, the expectation $F \to E(F) = F_0$ lies in every Σ_x. For a fixed x we have considerable freedom in constructing elements $[F]$ of A_x since we may arrange the behaviour of F for times exceeding T to be largely independent of its behaviour for times up until T. As an instance of this one can show that any sequence (x, T_n) of distinct characters arising from an increasing sequence of stopping times

$$T_0 \leqslant T_1 \leqslant T_2 \leqslant \cdots$$

is an interpolating sequence for A_x and so for $H_\infty(\mathscr{F})$.

The characters (x, t) for t a fixed time are not dense in Σ_x. To see this, let Q and Q' be the first exit times of Brownian motion from D and $2D$, respectively. The sets

$$\{\omega \in \Omega : |z_t(\omega)| < \tfrac{1}{2} \text{ and } t < Q(\omega)\}$$

for $0 \leqslant t < \infty$ form a filter base in $\mathscr{F}$ modulo the null sets so we can choose x to be an ultrafilter containing these. For $F = z^{Q'} \in H_\infty(\mathscr{F})$ we then have

$$|\hat{F}_t(x)| \leqslant \tfrac{1}{2} \qquad \text{for} \quad 0 \leqslant t < \infty,$$
$$|\hat{F}_\infty(x)| = |\hat{F}_{Q'}(x)| = 2$$

and

$$|\hat{F}_Q(x)| = 1.$$

So no point (x, t) for $0 \leqslant t \leqslant \infty$ can lie in the neighbourhood

$$\{\chi \in \operatorname{Spec} H_\infty(\mathscr{F}) : |\chi(F) - \hat{F}_Q(x)| < \tfrac{1}{2}\}$$

of (x, Q).

3. Analytic Structure in the Spectrum

In this section we will show that Spec $H_\infty(\mathscr{F})$ has no analytic structure, i.e., it contains no analytic discs. Stolzenberg [4] first exhibited a uniform algebra with this property, although our proof is closer in spirit to B. Cole's example (cf. [6]). Neither of these algebras was logmodular whereas we know that $H_\infty(\mathscr{F})$ is logmodular. Indeed, *every* real-valued random variable in $L_\infty(\mathscr{F})$ is of the form $\log|F|$ for some invertible $F \in H_\infty(\mathscr{F})$. Thus we see that even such stringent conditions as logmodularity are insufficient to ensure that the spectrum of an algebra has some analytic structure.

We begin by describing a functional calculus for $H_\infty(\mathscr{F})$. If f is a bounded function analytic on the image of $F \in H_\infty(\mathscr{F})$ then we find that $f \circ F$ is also in $H_\infty(\mathscr{F})$ (see [5, Lemma 3.1]). Our result extends this to germs of analytic functions in place of f. The full strength of the theorem is not needed for this section but it will form the basis of the subsequent one.

THEOREM 2. *Let R be a stopping time and $F \in H_\infty(\mathscr{F}_R)$. Suppose that ϕ is the germ of an analytic function at $F_0 = E(F)$ which can be continued analytically along almost every path*

$$F.(\omega)\colon t \to F_t(\omega) \qquad \text{for} \quad 0 \leqslant t < R(\omega)$$

giving a function $(\phi \cdot F)_t(\omega)$ with

$$|(\phi \cdot F)_t(\omega)| \leqslant K.$$

Then $\phi \cdot F$ is a holomorphic martingale in $H_\infty(\mathscr{F}_R)$.

Proof. By changing F on a null set we shall ensure that ϕ extends analytically along *every* path $F.(\omega)$. The derivatives $\phi^{(n)}$ will also do so, giving processes $\phi^{(n)} \cdot F$. If R' is a stopping time strictly less than R everywhere, then ϕ extends analytically along the *closed* path

$$F.(\omega)\colon t \to F_t(\omega) \qquad \text{for} \quad 0 \leqslant t \leqslant R'(\omega).$$

If we prove the theorem in this case for $F^{R'}$ then we can deduce the result in general. For we may choose $R'_n \nearrow R$ and obtain a martingale

$$E(\phi \cdot F \mid \mathscr{F}_{R'_n}) \in H_\infty(\mathscr{F}_R).$$

By the martingale convergence theorem, this converges almost everywhere to a holomorphic martingale $\phi \cdot F$. Now $|\phi \cdot F| \leqslant K$ almost surely so $\phi \cdot F \in H_\infty(\mathscr{F}_R)$ as required. Thus we may assume in our proof the ϕ extends along

$$F.(\omega)\colon t \to F_t(\omega) \qquad \text{for} \quad 0 \leqslant t \leqslant R(\omega).$$

Let Σ be the set of stopping times $T \leqslant R$ for which $(\phi \cdot F)^T$ is in $H_\infty(\mathscr{F})$ and $(\phi^{(n)} \cdot F)^T$ is previsible for each $n \in \mathbf{N}$. Then Σ is closed under the formation of countable suprema and it contains 0. Thus we can find $S \in \Sigma$ which maximizes, say, $E(\tan^{-1} S)$. Then any $T \in \Sigma$ must satisfy $T \leqslant S$ almost surely. To prove the theorem we shall show that $S = R$.

Consider those analytic functions $\theta: U \to \mathbf{C}$ defined on discs $U \subset \mathbf{C}$ and obtained by analytic continuation from ϕ. By passing to smaller discs we may assume that each U has rational centre and radius, while θ extends to a neighborhood of $\bar{U}$. The definition of analytic continuation ensures that there are only countably many such pairs (θ, U). For each one, let V be the set of ω for which $F_S(\omega) \in U$ and the germ of ϕ extended along $F.(\omega)$ to $F_S(\omega)$ agrees with θ, i.e.

$$V = \{\omega: F_S(\omega) \in U \,\&\, (\phi(n) \cdot F)_S(\omega) = \theta^{(n)}(F_S(\omega)) \text{ for } n \in \mathbf{N}\}.$$

Since $S \in \Sigma$, we have $V \in \mathscr{F}_S$ and each ω lies in one of the countably many sets V. If $P(S < R) > 0$ then

$$P(S(\omega) < R(\omega) \text{ and } \omega \in V) > 0$$

for some V. Now define

$$\begin{aligned} T(\omega) &= \inf(t \geqslant S(\omega): F_t(\omega) \notin U \text{ or } t = R(\omega)) \qquad \text{if} \quad \omega \in V \\ &= S(\omega) \qquad \text{otherwise.} \end{aligned}$$

This is a stopping time with $P(T > S) > 0$. Extend θ to a C_∞-function

$$\theta: \mathbf{C} \to \mathbf{C}.$$

Then Itô's lemma shows that $\theta \circ F$ is a stochastic integral with

$$d(\theta \circ F) = \frac{\partial \theta}{\partial z} \partial F\, dz + \frac{\partial \theta}{\partial \bar{z}} \overline{\partial F}\, d\bar{z} + \frac{1}{2} \nabla^2 \theta \;\cdot\; |\partial f|^2\, dt.$$

Because $\partial\theta/\partial\bar{z}$ and $\nabla^2\theta$ vanish at $F_t(\omega)$ for $S(\omega) \leqslant t < T(\omega)$, we find that

$$(\phi \cdot F)^S - (\phi \cdot F)^T = \theta \circ F^S - \theta \circ F^T$$

is a holomorphic martingale. Consequently, $(\phi \cdot F)^T$ is holomorphic and hence in $H_\infty(\mathscr{F})$. Similarly,

$$(\phi^{(n)} \cdot F)^S - (\phi^{(n)} \cdot F)^T = \frac{\partial^n \theta}{\partial z^n} \circ F^S - \frac{\partial^n \theta}{\partial z^n} \circ F^T$$

so $(\phi^{(n)} \cdot F)$ is previsible. This implies that $T \in \Sigma$ which contradicts the definition of S. So we must have $S = R$ almost surely. ∎

The above theorem, and its proof, may be adapted to the case of holomorphic martingales $F \in H_p(\mathscr{F}_R)$ for $p < \infty$.

THEOREM 3. *The spectrum $H_\infty(\mathscr{F})$ has no analytic structure.*

Proof. Hoffman [3] showed that the Gleason parts of a logmodular algebra, such as $H_\infty(\mathscr{F})$, were either single points or analytic discs. Suppose that there were an analytic disc. Then restriction to this disc gives an algebra homomorphism

$$\alpha\colon H_\infty(\mathscr{F}) \to H_\infty(D)$$

into the algebra of bounded analytic functions on the disc D. Let $F \in H_\infty(\mathscr{F})$ and $N \in \mathbf{N}$. If $E(F) \neq z_0$ then there is an analytic branch of $f\colon z \to (z-z_0)^{1/N}$ at $E(F)$. Also F is a time changed Brownian motion, so the paths $F.(\omega)$ almost surely avoid z_0. Thus f can be extended analytically along the paths (cf. Getoor and Sharpe [2; Theorem 5.6 and Lemma 5.7]). Theorem 2 now shows that there exists $G \in H_\infty(\mathscr{F})$ with $G^N = F - z_0 \cdot 1$. However, there must be some F for which the restriction $\alpha(F)$ is not constant. Then we can choose z_0 so that $\alpha(F) - z_0 \cdot 1$ has a zero in D. Since $\alpha(F) - z_0 = \alpha(G)^N$, this zero is of order a multiple of N for every natural number N. This is impossible unless $\alpha(F)$ is constant which was forbidden. ∎

Let Q be the first exit time from a domain D containing 0. Then Theorem 2 shows that an algebra embedding is defined by

$$M\colon H_\infty(D) \to H_\infty(\mathscr{F}); \qquad \phi \to \phi \cdot z^Q.$$

This induces a continuous map

$$\operatorname{Spec} M\colon \operatorname{Spec} H_\infty(\mathscr{F}) \to \operatorname{Spec} H_\infty(D).$$

When, for example, D is the disc, then this map is surjective. For if $z_0 \in D$ then we can readily find a stopping time $T \leqslant Q$ and a point $x \in \operatorname{Spec} L_\infty(\mathscr{F})$ with $\hat{z}_T(x) = z_0$. In this case $\operatorname{Spec} M$ maps (x, T) to the evaluation homomorphism at z_0. Carleson's corona theorem now shows that the image of M is the entire spectrum. A slight modification of this procedure shows that each fibre of $\operatorname{Spec} M$ is uncountable.

The map M was used by N. Th. Varopoulos [5] to deduce the corona theorem for $H_\infty(D)$ from his corresponding theorem for $H_\infty(\Omega)$. He also used the linear map

$$N\colon H_\infty(\mathscr{F}) \to H_\infty(D)$$

which sends F to the analytic function with boundary values $E(F \mid \mathscr{F}_Q)$. This satisfies $M \circ N = I$ but is not an algebra homomorphism. With the aid of Theorem 3 we can see that no algebra homomorphism with this property can exist. For if

$$L\colon H_\infty(\mathscr{F}) \to H_\infty(D)$$

were an algebra homomorphism, then

$$\text{Spec}\, L\colon \text{Spec}\, H_\infty(D) \to \text{Spec}\, H_\infty(\mathscr{F})$$

would map the Gleason part D into some Gleason part $\{\chi\}$ of $H_\infty(\mathscr{F})$. Hence L would be the trivial map

$$L\colon F \to \chi(F)\, 1_D.$$

4. Germs of Analytic Functions

The spaces $H_p(\mathscr{F})$ of holomorphic martingales are, in many respects, similar to the classical Hardy spaces $H_p(D)$. However, whereas the derivative of any analytic function is analytic, the derivative ∂F of a holomorphic martingale need not even be a martingale let alone holomorphic. It is therefore of some interest that one can construct spaces $H_p(A)$, intermediate between $H_p(D)$ and $H_p(\mathscr{F})$, which share the properties of $H_p(\mathscr{F})$ described in the previous section but also have all derivatives holomorphic. We shall briefly describe such a construction in this section, leaving the details to the interested reader. The construction is in terms of germs of analytic functions and displays an interesting link between Brownian motion and inner functions.

Consider the paths

$$t \to z_t(\omega) \qquad \text{for} \quad 0 \leqslant t < Q(\omega)$$

of Brownian motion from 0 to the boundary of the disc. We define A to be the set of analytic germs ϕ at 0 which can be continued analytically along almost every such path giving functions

$$t \to (\phi \cdot z)_t(\omega) \qquad \text{with} \quad |(\phi \cdot z)_t(\omega)| \leqslant K$$

for a constant K depending only on ϕ. This is clearly an algebra. Theorem 2 shows that we can embed it in each of the $H_p(\mathscr{F})$ spaces by

$$A \to H_p(\mathscr{F}); \qquad \phi \to (\phi \cdot z^Q).$$

Define $H_p(A)$ to be the norm closure of A in $H_p(\mathscr{F})$ for $1 \leqslant p < \infty$ and the weak* closure for $p = \infty$. For $\phi \in A$ we certainly have

$$\partial(\phi \cdot z^Q) = \phi' \cdot z^Q$$

a holomorphic martingale, so we see that the derivative ∂F of each $F \in H_p(A)$ is a holomorphic martingale. Thus the higher derivatives of F are defined and are also holomorphic. We shall be particularly concerned with $H_\infty(A)$ which is a Banach algebra.

(Note that the map

$$A \to L_\infty(\mathscr{F}); \qquad \phi \to (\phi \cdot z^Q)$$

embeds A as a non-closed uniform algebra on Spec $L_\infty(\mathscr{F})$. The probability P induces a representing measure for A on Spec $L_\infty(\mathscr{F})$ and the spaces $H_p(A)$ are precisely the abstract Hardy spaces for this representing measure. See, for example, [1, Chap. V].)

So far we have looked at $H_p(A)$ in terms of probability theory but to obtain its less immediate properties we need an alternative representation theorem for it in terms of analytic functions. To this end we prove the following theorem. Let $\mathscr{I}$ be the set of inner functions $f: D \to D$ which satisfy the normalization conditions:

$$f(0) = 0 \qquad \text{and} \qquad f'(0) \neq 0$$

Then each $f \in \mathscr{I}$ is locally invertible at 0 giving an analytic germ f^{-1} with $f^{-1}(0) = 0$. The set $\mathscr{I}$ is ordered by the relation

$$f \leqslant \tilde{f} \qquad \text{if } \exists g \in \mathscr{I} \qquad \text{with } \tilde{f} = f \circ g.$$

By using the Riemann mapping theorem, as in the proof below, we may show that $\mathscr{I}$ is then a directed set.

THEOREM 4. *The following conditions on an analytic germ ϕ at* 0 *are equivalent.*

(a) *$\phi \in A$, i.e., ϕ can be continued analytically along almost every path $t \to z_t(\omega)$ for $0 \leqslant t < Q(\omega)$ with $|\phi \cdot z_t(\omega)| \leqslant K$;*

(b) *there exists $f \in \mathscr{I}$ and $h \in H_\infty(D)$ with ϕ being the germ $h \circ f^{-1}$ at* 0 *and $\|h\|_\infty \leqslant K$.*

Proof. Suppose that (a) holds. Then we can construct a covering surface

$$\pi: (R, r_0) \to (D, 0)$$

and an analytic map $\Phi: R \to \mathbf{C}$ such that almost every path $t \to z_t(\omega)$ lifts to a path $t \to \tilde{z}_t(\omega)$ in R starting from the base point r_0, and having $\phi \cdot z_t(\omega) = \Phi(\tilde{z}_t(\omega))$ for $0 \leqslant t < Q(\omega)$. We can even choose Φ with $\|\Phi\|_\infty \leqslant K$. Since π is a non-constant analytic map, the universal cover of R cannot be $\mathbf{C}$ or $\mathbf{C} \cup \{\infty\}$. The Riemann mapping theorem implies that it is D so we may find a universal covering map $\psi: (D, 0) \to (R, r_0)$. Now (b) follows with $f = \pi \circ \psi$ and $h = \Phi \circ \psi$.

Conversely, suppose that (b) holds. Consider the bounded holomorphic martingale $f \cdot z^Q$ constructed by Theorem 2. It is a time change of some new Brownian motion, say w. In particular,

$$fz_Q(\omega) = w_T(\omega)$$

for some stopping time T. The maximum modulus principle shows that

$$|fz_t(\omega)| < 1 \qquad \text{for} \quad 0 \leqslant t < Q(\omega)$$

so

$$|w_s(\omega)| < 1 \qquad \text{for} \quad 0 \leqslant s < T(\omega).$$

Since f is inner, the non-tangential limits of f have absolute value 1 almost everywhere. Therefore $|fz_Q(\omega)| = 1$ almost surely. These observations imply that

$$T(\omega) = \inf(s\colon |w_s(\omega)| \geqslant 1),$$

so w^T, like z^Q, is a Brownian motion starting at 0 and stopped on leaving D. In proving (a) we may therefore replace z by w.

By construction, f^{-1} extends analytically along the path $s \to w_s(\omega)$ for $0 \leqslant s < T(\omega)$ giving the path $t \to z_t(\omega)$ for $0 \leqslant t < Q(\omega)$. Since h is analytic on D, we deduce that $0 = h \circ f^{-1}$ can be continued along $s \to w_s(\omega)$ for $0 \leqslant s < T(\omega)$ with $|\phi \cdot w_s(\omega)| \leqslant \|h\|_\infty \leqslant K$. ∎

In view of this theorem we can give a purely analytic definition of A. For $f \in \mathscr{I}$ let $H_p(D_f)$ be a copy of $H_p(D)$ and define M_f to be the map

$$M_f\colon H_p(D_f) \to H_p(\mathscr{F}),$$
$$h \to h \circ (f^{-1} \cdot z^Q).$$

For f the constant function 1, this is simply the map M studied in [5]. When $f \leqslant \tilde{f}$ we define

$$M_{\tilde{f}f}\colon H_p(D_f) \to H_p(D_{\tilde{f}}),$$

by

$$f \to h \circ g,$$

where $g \in \mathscr{I}$ is the function with $\tilde{f} = f \circ g$. The spaces $H_p(D_f)$ with these maps form a direct system with the maps M_f as the canonical maps into the direct limit. Thus the theorem allows us to identify A as the direct limit of the spaces $H_\infty(D_f)$ which, in this case, is simply the union $\bigcup M_f(H_\infty(D_f))$. Similarly, the spaces $H_p(A)$ are appropriate completions of such direct limits.

When $p = \infty$, observe that each M_f is a weak* continuous algebra homomorphism.

We can extend M_f to the map

$$M_f\colon L_p(T) \to L_p(\mathscr{F}),$$
$$h \to h \circ (f^{-1} \cdot z^Q).$$

This maps onto $L_p(\mathscr{S}(f))$, where $\mathscr{S}(f)$ is the σ-field generated by the *single* random variable $f^{-1} \cdot z^Q$ which almost surely takes values on the unit circle $\mathbf{T}$. Thus, if $\mathscr{S}$ is the σ-field generated by $\mathscr{S}(f)$ for all $f \in \mathscr{I}$, then $H_p(A)$ is a subspace of $L_p(\mathscr{S})$. In fact much more is true and $L_p(\mathscr{S})$ stands in the same relation to $H_p(A)$ as $L_p(\mathbf{T})$ does to $H_p(D)$ or $L_p(\mathscr{F})$ to $H_p(\mathscr{F})$.

The duals of the mapping $M_f\colon L_p(\mathbf{T}) \to L_p(\mathscr{F})$ for $1 \leqslant p < \infty$ are continuous linear mappings

$$N_f\colon L_q(\mathscr{F}) \to L_q(\mathbf{T})$$

for q with $1/p + 1/q = 1$. Moreover, these map $H_q(\mathscr{F})$ into $H_q(D_f)$. The composite $M_f \circ N_f$ is the conditional expectation $E(F \mid \mathscr{S}(f))$ while $N_f \circ M_f$ is the identity. Now consider the conditional expectation

$$E(\ \mid \mathscr{S})\colon L_q(\mathscr{F}) \to L_q(\mathscr{S}).$$

We claim that this maps $H_q(\mathscr{F})$ into $H_q(A)$. For suppose that $F \in H_q(\mathscr{F})$ and $G = E(F \mid \mathscr{S})$. Then G is measurable relative to the σ-field generated by $\mathscr{S}(f_n)$ for some *countable* sequence $f_n \in \mathscr{I}$. Moreover we can assume that f_n, and hence $\mathscr{S}(f_n)$, increase with n. For each n, $E(F \mid \mathscr{S}(f_n)) = E(G \mid \mathscr{S}(f_n))$ is in $M_{f_n}(H_q(D_{f_n}))$ because N_{f_n} maps $H_q(\mathscr{F})$ into $H_q(D_{f_n})$. The martingale convergence theorem implies that $E(G \mid \mathscr{S}(f_n)) \to G$ as $n \to \infty$ (weak* if $q = \infty$). Thus $G \in H_q(A)$ as claimed. We have now constructed linear maps

$$H_q(D_f) \xrightarrow{M_f} H_q(A) \longrightarrow H_q(\mathscr{F})$$

and splitting maps

$$H_q(D_f) \xleftarrow{N_f} H_q(A) \xleftarrow{E(\ \mid \mathscr{S})} H_q(\mathscr{F}).$$

These enable us to deduce the properties of $H_q(A)$ from those of $H_q(D)$ and $H_q(\mathscr{F})$ in a way similar to that of N. Th. Varopoulos in [5].

For example, if u is a real-valued random variable in $L_2(\mathscr{S})$, then it certainly lies in $L_2(\mathscr{F})$ and so has an unique conjugate function $\tilde{u} \in L_2(\mathscr{F})$ with $u + i\tilde{u} \in H_2(\mathscr{F})$. Now

$$u + iE(\tilde{u} \mid \mathscr{S}) = E(u + i\tilde{u} \mid \mathscr{S}) \in H_2(\mathscr{S})$$

so $\tilde{u} \in L_2(\mathscr{S})$ and the conjugation operator on $L_2(\mathscr{F})$ restricts to give the conjugation operator $L_2(\mathscr{S}) \to L_2(\mathscr{S})$. If $u \in L_\infty(\mathscr{S})$ then $u + i\tilde{u} \in H_2(A)$ and Theorem 2 shows that $F = \exp(u + iu)$ is an invertible element of $H_\infty(\mathscr{F})$. Since F is also $\mathscr{S}$-measurable, it is an invertible element of $H_\infty(A)$. In particular $H_\infty(A)$ is a logmodular algebra on Spec $L_\infty(\mathscr{S})$. The argument of Section 3 also applies to $H_\infty(A)$ showing that its spectrum has no analytic structure.

The maps constructed above are also sufficient to allow one to deduce from the corona theorem for $H_\infty(\mathscr{F})$ the corresponding theorem for $H_\infty(A)$, in exactly the same way that the classical corona theorem for $H_\infty(D)$ was deduced in [5].

References

1. T. W. Gamelin, "Uniform Algebras," Prentice-Hall, Englewood Cliffs, N. J., 1969.
2. R. K. Getoor and M. J. Sharpe, Conformal martingales, *Invent. Math.* **16** (1972), 271–308.
3. K. Hoffman, Analytic functions and logmodular Banach algebras, *Acta Math.* **108** (1962), 271–315.
4. G. Stolzenberg, A hull with no analytic structure, *J. Math. Mech.* **12** (1963), 103–111.
5. N. Th. Varopoulos, The Helson–Szegö theorem and A_p-functions for Brownian motion and several variables, *J. Funct. Anal.*, in press.
6. J. Wermer, "Banach Algebras and Several Complex Variables," Graduate Texts in Mathematics 35, 2nd ed., Springer-Verlag, Berlin/New York, 1976.